

# Geometric Approach to Chebyshev Net Generation Along an Arbitrary Surface Represented by NURBS

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## Abstract

Geometric and computational approach based on the theory of Chebyshev nets is developed for the analysis of an arbitrary surface dressed by woven fabric or for the construction of the fiber trajectories in each lamina of braiding reinforcement in laminated composite double curved shells. The approach is also applicable to fabric tension structure cutting pattern generation. The efficient computer tool is developed on the basis of the approach described in this paper, and is included into FABRIC CAD system aimed at fabric tension structures design.

**Key words:** *Fabric tension structure, Chebyshev nets, laminated composite double curved shells, NURBS representation, FABRIC CAD system.*

## 1. INTRODUCTION

Over the past 20 years, there has been a rapid growth in the use of fabric tension structures. A fabric tension structure has the advantage of covering very large areas with clear spans and provides the ideal solution for such applications as indoor tennis courts and swimming pools. It also offers low construction and maintenance costs, high mobility and prefabrication possibilities. Further, such structures, with doubly-curved shapes, are aesthetically pleasing. The design of a fabric tension structure, however, is a complex task. Three steps are usually required in the design process of tension structure, *namely, form finding, load analysis and cutting pattern generation*. Form finding problem is mostly performed by first establishing a mesh over a surface and may be resolved by some formulations of FEM or by the so called Stretched Grid Method (SGM) as it is described in [1], [2]. Once a satisfactory shape has been found, a cutting pattern may be generated. Tension structures are highly varied in their size, curvature and material stiffness. Cutting pattern approximation is strongly related to each of these factors. It is essential for a cutting pattern generation method to minimize possible approximation and to produce reliable plane cloth data. The objective is to develop the shapes described by these data, as near as possible to the ideal doubly-curved strips. In general, cutting pattern generation involves two steps. First, the global surface of a tension structure is divided into individual cloth layout. The corresponding cutting pattern at the second step can be found by simply taking each cloth strip and unfolding it on a planar area. In the case of the ideal doubly-curved membrane surface the subsurface cannot be simply unfolded and they must be flattened. In [3] the flattening problem has been resolved by the form of SGM.

As well as fabric tension structures the laminated shell structures laminated by fibre arrays, such as 3-D Sails etc, are very popular. Typically, design of a laminated shell is based on a strength analysis at one or several critical locations over the shell surface. At one such location, the layout may be optimised (e.g., by minimizing the amount of reinforcement) subject to specified strength constraints. However, a given directional lay-up of the fabric, braiding or unidirectional tape reinforcement is, generally, realiz-

able only at one location in the shell; elsewhere, the orientation of fibre arrays in each lamina must vary according to the shell geometry. The extent of the variation depends on the Gaussian curvature of the surface and, for a doubly curved shell, the reinforcement at some locations may turn out to be statically inadequate or even geometrically infeasible.

At the current state of the art in the manufacturing of doubly curved laminated shells, the reinforcement flat cutting patterns for each lamina are often obtained experimentally, which is both time-consuming and costly. Moreover, even when aided by computer, the user has little control over the geometric properties of the required cutting patterns.

A comprehensive geometric analysis of a single lamina is developed in a form suitable for subsequent geometric optimisation of a laminate layout and is described in [4]. This paper is dedicated to developing of the efficient means for cutting pattern problem solution. The means are based on Chebyshev net theory.

## 2. THE SETTING OF THE PROBLEM

Consider the problem of how to generalize the notion of "Cartesian grid" (chess board) on a general, curved surface. Clearly one has to do some concessions in order to be able to apply the net to the surface. Two obvious possibilities are, first, to keep all angles at  $\pi/2$ , or, second, to preserve the equality of edge lengths. In the first case one obtains conformal nets. In general their edge lengths will vary from place to place. The nets of principal curvature directions are one possibility (since the principal directions are orthogonal), but infinitely many others exist. In the second case one obtains the so-called Chebyshev nets. In that case the angles vary from place to place, i.e., the mazes become parallelograms. Further we consider this second possibility.

The theory of Chebyshev nets is to be of interest among the surface internal geometry problems. In 1878 P.L. Chebyshev has presented the report "On the cutting of our clothes" [5], where he formulated some rules of great practical and theoretical meaning. For cutting out of clothes, as well as for manufacturing of fabric shells for various structures, the problem is to make a planar pattern on a fabric dressed a curvilinear surface that might fit it in a possibly better way.

It was shown by Chebyshev, that in that case there can be no surface bending, because the fabric and surface fitness can be obtained by the network distortion in diagonal direction of an interlacing rather than of a fabric interlacing fibers stretching. At the first assumption Chebyshev considered fabric-interlacing fibers were inextensible. When one dresses a surface by a fabric of a classical interlacing, which consists of two generally orthogonal fibers directions called warp and weft, the curvilinear network over a surface is formed. As a result the network corners are changed but the fibers are bended with their lengths being conservative. Such network has the following properties: two lines belonging to one family of the fibers pick out arches of the same

length at any string belonging to a family of other strings. In other words, in any curvilinear rectangle, formed by lines of Chebyshev net, the opposite edges are equal.

Chebyshev's idea mentioned above was the basis for the mathematical approach that allows one the developing of a planar rectangular net mapping with the conservative cell segment length. The approach is based on decomposition of the Chebyshev net coordinate lines at semi-geodetic net coordinates.

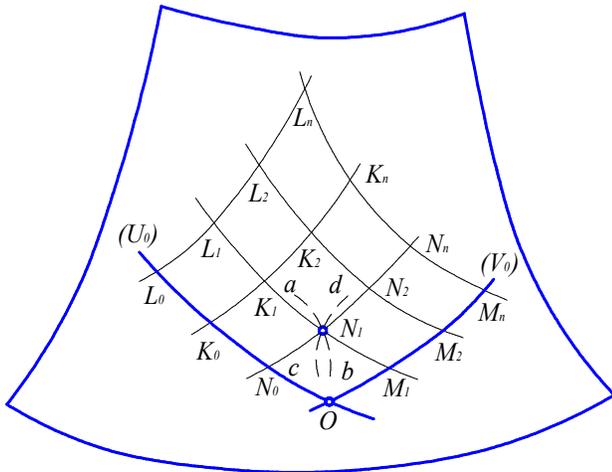


Figure 1: The surface area near regular point  $O$ .

The general problem is to develop Chebyshev net on an arbitrary double-curved surface with known shapes. It is known [6], that there is an uncountable set of Chebyshev nets on a planar surface. Each Chebyshev net is completely defined by two lines belonging to the first, and second fiber families. The lines can set across a point arbitrary but in a various directions. Naturally, it is possible to develop Chebyshev nets randomly at any surface. Let us show it by the following sample (See Fig. 1). Let two arbitrary lines  $(U_0, V_0)$  in different directions on surface  $S$  plotted through a regular point  $O$ . Let us divide each line into series of equal arches with length  $\rho$  (along  $U_0$ ) and  $\sigma$  (along  $V_0$ ). Draw arch  $ab$  of geodetic circle with the center at point  $M_1$  and with Gaussian radius  $\rho$ . Draw a similar geodetic circle arch  $cd$  of Gaussian radius  $\sigma$  with the center at point  $N_0$ . The current point  $N_1$  that one can consider as a Chebyshev net vertex may be obtained at the point of two arches  $ab$  and  $cd$  intersection. Other vertices, i.e.  $N_2, \dots, N_n, K_1, \dots, K_n, M_1, \dots, M_n$ , may be obtained by a similar drawing.

It may be shown that the net obtained by the described above drawing converges to the exact Chebyshev net simultaneously with the convergence  $\rho$  and  $\sigma$  to zero. Obvious that the direction of two lines  $U_0$  and  $V_0$  can be definitely arbitrary (excluding the case of coincidence). Then one may develop an uncountable set of Chebyshev net using the same initial point  $O$  of an arbitrary surface. Besides one may deduce that a very convenient approximation approach to Chebyshev net generation at an arbitrary surface may be developed.

### 3. THE APPROXIMATE APPROACH

Any modern geometry design systems, analysis of geometric objects, conversion from one geometric representation to another and graphical display of geometric objects, if they are to work well, must be based on sound mathematical theories and numerical methods. If the underlying computation and mathematics are sound, one can expect systems based on them to, by and large,

perform well, be easy to maintain and be adaptable to new, currently unforeseen, needs as they arise. For geometry design, it is generally understood that methods based on splines using the B-spline, as well as its further advancement like NURBS representation, provide the best tools for meeting these goals. Taking it into account the NURBS surface representation is added to mathematical arsenal of FABRIC CAD system (see for example [7], [8], [9]), aimed at tension fabric structures design.

The key fabric structures design problem is to generate cutting pattern of each double-curved fabric cloth. FABRIC CAD system is provided with a special procedure based on SGM as it is described in [7], [9] that allows a designer to flatten double-curved patches of shells. If the fabric structure cloth is manufactured from fabric of traditional interweaving it seems more preferable to use another approach based on the Chebyshev net properties. The approach, however, provides for an availability of the Chebyshev net with desirable attributes that covers given and in general arbitrary surface represented by NURBS. Formally, construction of a Chebyshev net on a given surface calls for solving an initial value problem for a simultaneous system of two nonlinear partial differential equations. The structure of the equations rules out the possibility of an analytical solution even for simple surfaces (with the exception of a few classical closed-form solutions, such as translation nets on translation surfaces); thus, the only practical course of action is to solve the problem approximately. The developed approximate procedure must be computationally very efficient. Such approximated approach is developed on the basis of idea outlined in previous section and is used by special procedure available in FABRIC CAD toolbox. After comparing a few possible approaches, a cell-by-cell construction of a Chebyshev net emerged as the most promising method. The method capitalizes on the characteristic property of the net—the rhombic shape of its elementary cell.

At the heart of the approach is a “Spherical Tool” concept that is introduced into lexicon to explain how one may produce a geodetic circle near given point of an arbitrary surface. The Spherical Tool is a usual spherical surface represented by NURBS with predefined radius that a designer could change as he wishes according to his needs. Placing the center of Spherical Tool at the desirable surface point one may obtain a geodetic circle as a NURBS line of two NURBS surfaces intersection, i.e., the original surface and spherical tool. Obviously, in such case a geodetic circle is a closed regular line that can be represented both in 3D Cartesian and 2D surface parametric spaces. Placing further the spherical tool at another surface point one may obtain another geodetic circle. Both geodetic circles may be treated as two closed planar curves in 2D surface parameter space so it is very easy to calculate their points of intersection.

Definitely the problem of two NURBS surfaces intersection must be developed well. Planar cut and surface intersection software is an important part of any CAD system. A number of different fundamental approaches to this problem are followed in these systems, including geometric methods for quadric sections, subdivision-based methods, implicit schemes and curve tracing schemes etc. The latter schemes are often referred to as “marching” or “curve following” schemes in the literature. In recent years, “loop detection” has become the hot topic in this field. The technique is described in [10] so will not be discussed here.

FABRIC CAD system provides a designer with a possibility to develop Chebyshev net dressed an arbitrary NURBS surface pressing one button only. Let us demonstrate it by the following

sample (See Fig. 2). The Chebyshev net development comes in the following steps:

1. The initial surface point  $O$  is defined. Naturally the initial point may be arbitrary. As a default point in FABRIC CAD system is considered the point at the middle of 2D surface parameter space. However, the user may define another point.

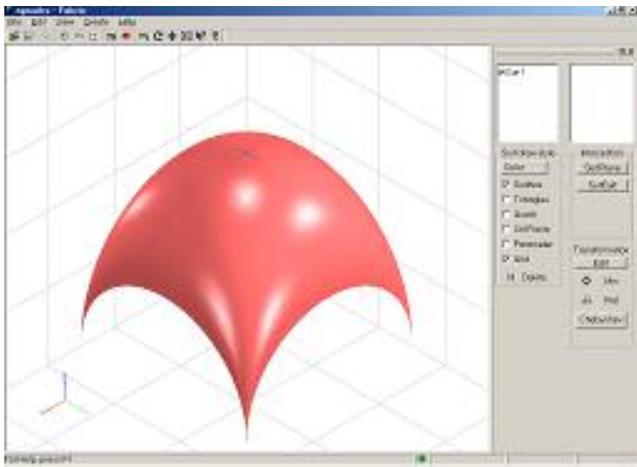


Figure 2: The given surface represented by NURBS.

2. The centre of spherical tool of desirable radius is located to a given point (See Fig. 3) and two NURBS surfaces are intersected to find a parametric representation of a geodetic circle. Note that the radius of spherical tool may vary according to size of final Chebyshev net cells. The default value of radius is equal to 1/25 of the length of shortest NURBS surface edge.

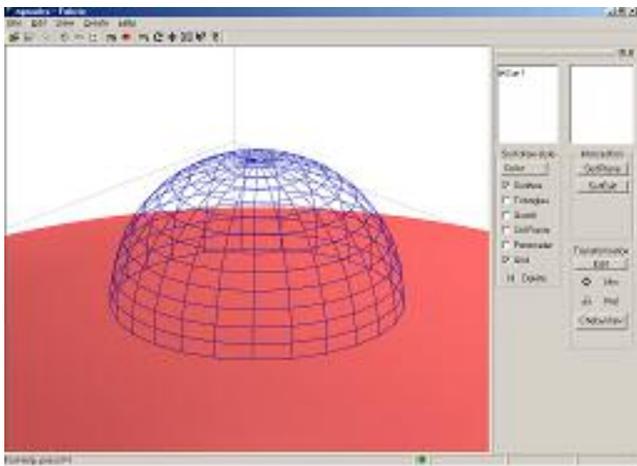


Figure 3: Spherical NURBS tool placed at the initial surface point

3. Two unmatched surface directions  $T_1$  and  $T_2$  at the given point  $O$  are defined. Since such directions may differ, two 2D parameter lines  $U_o = const$ ,  $V_o = const$  (where  $U, V$  – surface parameters) are selected as a default directions. The user may define two other directions. Reasonably the shape of final Chebyshev net will depend on it.

4. The geodetic circle defined at the step 2 is intersected with parametric lines  $U_o = const$ , and  $V_o = const$  to obtain for points  $1, 2, 3, 4$  (See Fig. 4) that combine into the “initial cross” shown in Fig. 5 in 3D Cartesian space.

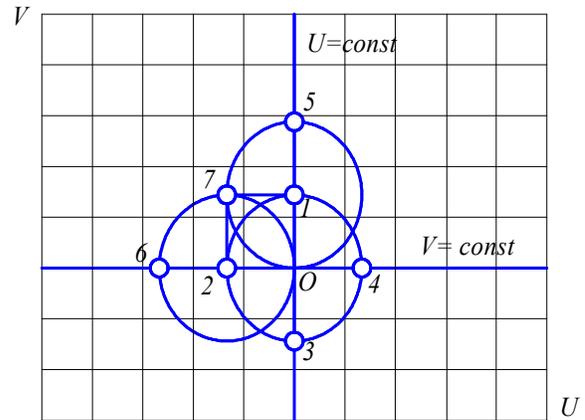


Figure 4: Generation of Chebyshev net vertices in 2D space.

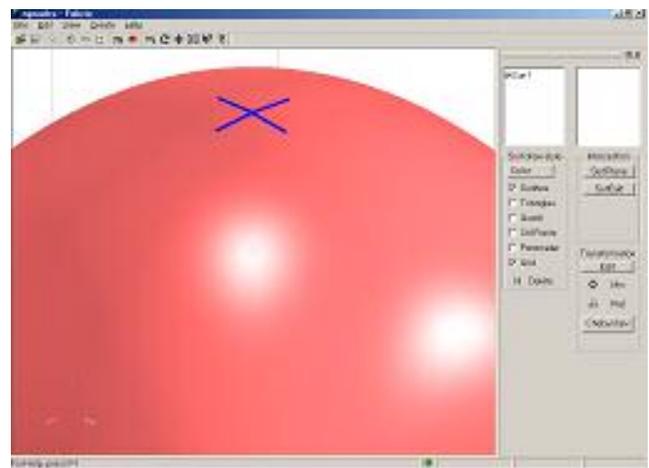
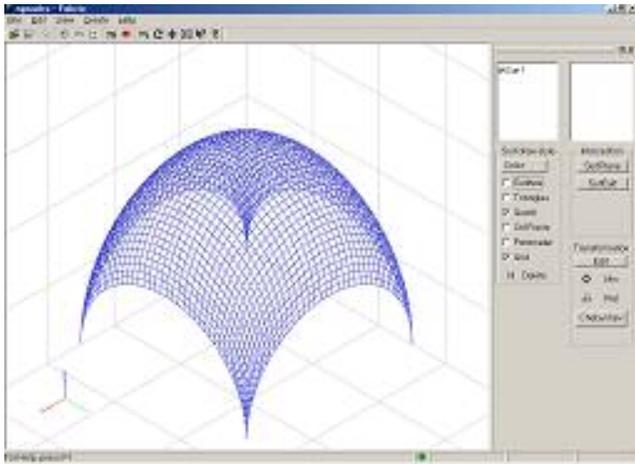


Figure 5: “The initial cross” at the beginning of Chebyshev net.

5. We shift the centre of spherical tool to points  $1, 2, 3, 4$  in turns and intersect produced geodetic circles with lines  $U_o = const$ , and  $V_o = const$  and between each other in pairs such as it is shown in Fig. 4 to generate Chebyshev net vertices  $5, 6, 7$  completing cells and so on. The cycle is repeated “spiralwise” around point  $O$  in ascending direction as long as the whole surface area is covered by Chebyshev net vertices. Thus, completing a cell with already located vertices represents a recurring generic step of the net construction procedure. It should be noted that problems arise at the surface edges, where some current vertices become imaginary, i.e., lie out of the surface parameter space. Moreover, it is impossible to compute two geodetic circles intersection here. In such case the system produce current vertex by simple extrapolation using previous vertices. The final Chebyshev net covering the given surface is presented in Fig. 6

## 4. CONCLUSION

The developed fast and computationally efficient geometric generation of the Chebyshev net can be useful for a variety of problems, from general CAD-like application for surface “clothing” and fabric tension structures design to computer aided design of laminated composite shells.



**Figure 5:** The final Chebyshev net dressed a given surface.

Taking the initial directions of two intersecting fibers (directional lay-up) at some point on the surface, one may determine the two respective reference fiber trajectories on the surface in the engendered Chebyshev net. A designer may vary such trajectories varying the orientation of the initial cross at the beginning of the net.

At the same time two difficulties with the described approach are immediately obvious. The first one, rooted in the very nature of the problem as the initial value problem, is the solution sensitivity both to initial conditions (to location and orientation of an initial cross) and to the accumulation and propagation of computational error. The second difficulty is associated with the cell-by-cell net generation method itself, even assuming a modest computational effort involved in each cell completion step. The fact is that for the purpose of generation, the elementary cells, theoretically infinitesimal in size, must be aggregated into finite size curvilinear rhombuses. To maintain reasonable accuracy, the aggregated cell should be sufficiently small, hence, a large number of cells is needed. The major challenge is a very large number of cell completion steps in the overall scheme. Thus, these problems need further studying.

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