Edge Detection Using Reaction-Diffusion Equation with Variable Diffusion Coefficient

Alexander M. Denisov, Andrey S. Krylov, Victoria Yu. Medvedeva
Faculty of Computational Mathematics and Cybernetics
Lomonosov Moscow State University
den@cs.msu.ru , kryl@cs.msu.ru, medvedevav_88@mail.ru

Abstract

New edge detection method using FitzHugh-Nagumo reaction-diffusion system with image-dependent variable diffusion coefficient is suggested. It avoids false edges arising for noisy images and images with ringing in the case of commonly used constant diffusion coefficient.

Keywords: *edge detection, FitzHugh-Nagumo model, variable diffusion coefficient*

1. INTRODUCTION

Diffusion filtering methods are an effective tool for edge detection. These methods usually use as a criterion the zero-crossing of the second partial derivatives. As an example, Marr-Hildreth operator is based on the zero-crossing analysis of image convolution with the Laplacian of the Gaussian function(LoG). An alternative edge detection method uses the difference of two Gaussian filters with different diffusion coefficients(DoG, also introduced by Marr and Hildreth) (see review in [1]).

The Gaussian filter is the solution of the diffusion equation. Thus the computation of the difference of two Gaussian filters can be replaced by a numerical solution of the system of two diffusion equations with different diffusion coefficients. This replacement idea is a basis of many filtering algorithms like anisotropic diffusion.

The idea to use the FitzHugh-Nagumo reaction-diffusion system of equations for binary image edge detection was suggested in [2] by Nomura et al. Later they suggested a modification of this algorithm for greyscale images: local threshold value method [3]. Nevertheless, these methods produce false edges for noisy images and images with ringing artifact. In this paper, to overcome this problem we propose to use the FitzHugh-Nagumo reaction-diffusion system with image-dependent variable diffusion coefficient.

2. FITZHUGH-NAGUMO MODEL

The FitzHugh-Nagumo system can be represented as follows:

$$\frac{\partial u}{\partial t} = D_u \Delta u + f(u,v), \Omega \times [0,T],$$
$$\frac{\partial v}{\partial t} = D_v \Delta v + g(u,v), \Omega \times [0,T],$$
$$f(u,v) = \frac{1}{\varepsilon}(u(1-u)(u-a)-v),$$
$$g(u,v) = u - bv,$$
$$u(x,t,t=0) = I(x,y), \Omega,$$
$$v(x,y,t=0) = 0, \Omega,$$
$$\frac{\partial u}{\partial n}|_{\partial\Omega} = \frac{\partial v}{\partial n}|_{\partial\Omega} = 0,$$

where Ω is rectangle $[0,x_0] \times [0,y_0]$, ε is a small positive constant, D_u and D_v are positive constants $D_u < D_v$.

If we want to understand the time behaviour of u and v function, we can consider a simplified system:

$$\frac{\partial u}{\partial t} = f(u,v), \Omega \times [0,T],$$
$$\frac{\partial v}{\partial t} = g(u,v), \Omega \times [0,T],$$
$$f(u,v) = \frac{1}{\varepsilon}(u(1-u)(u-a)-v),$$
$$g(u,v) = u - bv.$$

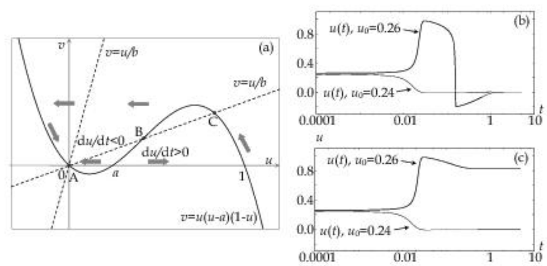

Figure 1. FitzHugh-Nagumo model behavior. (a)Phase plot. (b)monostable system(a=0.25,b=1). (c)bistable system(a=0.25,b=10). $\varepsilon = 10^{-3}$ [2]

Picture 1(a) shows that function u increases in the area below the $v = u(u-1)(a-u)$ curve, and function v increases in the area below $u = bv$ line. Thus, if our starting point is $(u,0), u \in (a,1)$ and the curves intersect, the system will achieve an equilibrium in the C point. If we start from $(u,0), u \in (0,a)$ point, the system will achieve an equilibrium in the A point. If lines do not intersect, the system will achieve an equilibrium in the A point in both cases. Thus, the behaviour of the system depends on ratio of the a and b parameters. It can be monostable or bistable.

Full equation system causes appearance of the pulses in the edge points. FitzHugh-Nagumo system can detect edges of the binary image, but it is impossible for it to detect edges of greyscale image, because the FitzHugh-Nagumo system defines edge using global threshold value. Local threshold value was introduced for the purpose of greyscale images edge detection [3].

Problem with local threshold value

$$a = a(x,y,t),$$

can be represented as follows:

$$
\begin{gathered}
\frac{\partial u}{\partial t} = D_u \Delta u + f(u,v), \Omega \times [0,T], \\
\frac{\partial v}{\partial t} = D_v \Delta v + g(u,v), \Omega \times [0,T], \\
\frac{\partial a}{\partial t} = D_a \Delta a, \Omega \times [0,T], \\
f(u,v) = \frac{1}{\varepsilon}(u(1-u)(u-a)-v), \\
g(u,v) = u - bv, \\
u(x,t,t=0) = a(x,t,t=0) = a_0 * I(x,y), \Omega\,, \\
v(x,y,t=0) = 0, \Omega, \\
\frac{\partial u}{\partial n}|_{\partial\Omega} = \frac{\partial v}{\partial n}|_{\partial\Omega} = \frac{\partial a}{\partial n}|_{\partial\Omega} = 0.
\end{gathered}
$$

Diffusion coefficient D_a should satisfy $D_a > D_v$ condition and $0 < a_0 < 1$.

3. FITZHUGH-NAGUMO MODEL WITH VARIABLE DIFFUSION COEFFICIENT

FitzHugh-Nagumo system with local threshold value allows us to make an effective edge emphasis for "clear" images. But false edges appear in the case of noise or ringing artefact (Gibbs effect). To solve this problem we introduce the variable diffusion coefficient, as in Perona-Malik approach[4]:

$$h(x,y) = \frac{1}{1 + \lambda^2 |\bigtriangledown I(x,y)|^2}.$$

The suggested diffusion coefficient is small in the edge areas, and diffusion in smooth areas is accelerated. These areas will be denoised more intensively. Thus we do not have false edges that can appear in the case of constant diffusion coefficient.

Problem with variable diffusion coefficient can be represented as follows:

$$
\begin{gathered}
\frac{\partial u}{\partial t} = div(h(x,y) \bigtriangledown u) + f(u,v), \Omega \times [0,T], \\
\frac{\partial v}{\partial t} = D_v \Delta v + g(u,v), \Omega \times [0,T], \\
\frac{\partial a}{\partial t} = D_a \Delta a, \Omega \times [0,T], \\
f(u,v) = \frac{1}{\varepsilon}(u(1-u)(u-a)-v), \\
g(u,v) = u - bv, \\
u(x,t,t=0) = a(x,t,t=0) = a_0 * I(x,y), \Omega, \\
v(x,y,t=0) = 0, \Omega, \\
\frac{\partial u}{\partial n}|_{\partial\Omega} = \frac{\partial v}{\partial n}|_{\partial\Omega} = \frac{\partial a}{\partial n}|_{\partial\Omega} = 0.
\end{gathered}
$$

4. INFLUENCE OF VARIABLE DIFFUSION COEFFICIENT ON THE QUALITY OF EDGE DETECTION

The results of the application of edge detection algorithm with variable diffusion coefficient are represented in figures 2-9. The algorithm has been applied to noisy images and images with ringing. For comparison the results of constant diffusion coefficient algorithm with $D_u = 1.5$ are given.

The following difference scheme was chosen for the computation:

$$
\begin{gathered}
\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\tau} = h(i,j)\Big(\frac{u^n_{i-1,j} - 2u^n_{i,j} + u^n_{i+1,j}}{(\triangle x)^2} + \\
+ \frac{u^n_{i,j-1} - 2u^n_{i,j} + u^n_{i,j+1}}{(\triangle y)^2}\Big) + \frac{(u^n_{i,j} - u^n_{i-1,j})(h(i,j) - h(i-1,j))}{(\triangle x)^2} + \\
+ \frac{(u^n_{i,j} - u^n_{i,j-1})(h(i,j) - h(i,j-1))}{(\triangle y)^2} + f^n_{i,j},
\end{gathered}
$$

$$
\begin{gathered}
\frac{a^{n+1}_{i,j} - a^n_{i,j}}{\tau} = \\
= D_a\Big(\frac{a^n_{i-1,j} - 2a^n_{i,j} + a^n_{i+1,j}}{(\triangle x)^2} + \frac{a^n_{i,j-1} - 2a^n_{i,j} + a^n_{i,j+1}}{(\triangle y))^2}\Big),
\end{gathered}
$$

$$
\begin{gathered}
\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\tau} = \\
= D_v\Big(\frac{v^n_{i-1,j} - 2v^n_{i,j} + v^n_{i+1,j}}{(\triangle x)^2} + \frac{v^n_{i,j-1} - 2v^n_{i,j} + v^n_{i,j+1}}{(\triangle y)^2}\Big) + g^n_{i,j},
\end{gathered}
$$

$$f^n_{i,j} = \frac{1}{\varepsilon}(u^n_{i,j}(1 - u^n_{i,j})(u^n_{i,j} - a_{i,j}) - v^n_{i,j}),$$

$$g^n_{i,j} = u^n_{i,j} - bv^n_{i,j},$$

$$h(i,j) = \frac{1}{1 + \lambda^2((\frac{u^n_{i,j} - u^n_{i-1,j}}{\triangle x})^2 + (\frac{u^n_{i,j} - u^n_{i-1,j}}{\triangle y})^2)}.$$

The parameters are: $a_0 = 0.25$; $b = 5$; $D_v = 5$; $D_a = 10$; $\varepsilon = 0.001$; $\lambda = 1$.

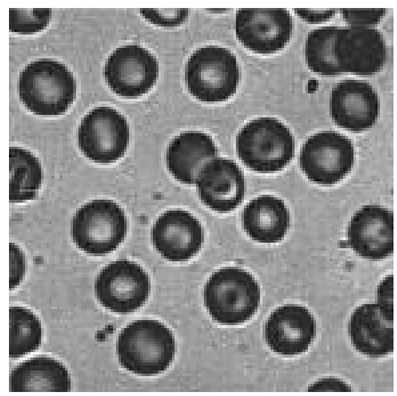

Figure 2. Source noisy image

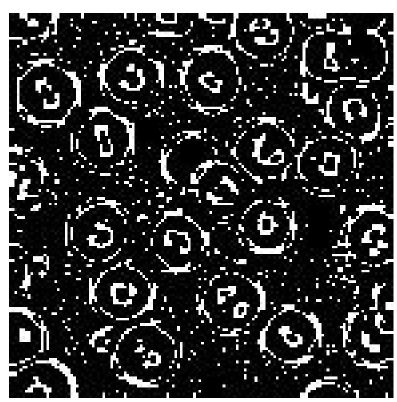

Figure 3. Result of constant diffusion coefficient FitzHugh-Nagumo edge detector

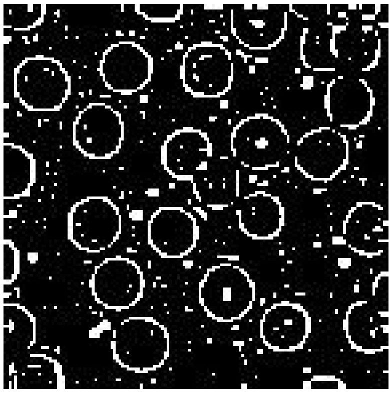

Figure 4. Result of FitzHugh-Nagumo edge detector with variable diffusion coefficient

As we see in the figures 2 - 4 , for noisy images FitzHugh-Nagumo model with variable diffusion coefficient detects edges more effectively . It is also effective for images with ringing and noise. The results with the same parameters of computation for images with ringing are shown in the figures 5 - 7, for images with noise and ringing - in figures 8 - 10.

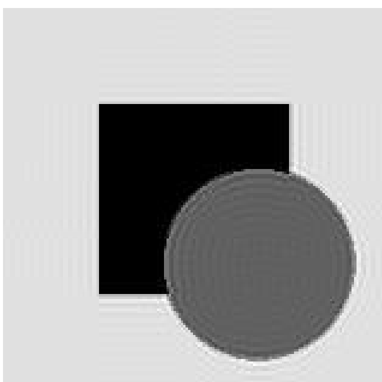

Figure 5. Source image with ringing

5. CONCLUSION

It was shown that the introduction of the image dependent variable coefficient makes edge detection more effective and allows to avoid false edges, caused by noise and ringing. The work was supported by federal target program Scientific and scientific-pedagogical personnel of innovative Russia in 2009-2013 and RFBR grant 08-01-00314.

6. REFERENCES

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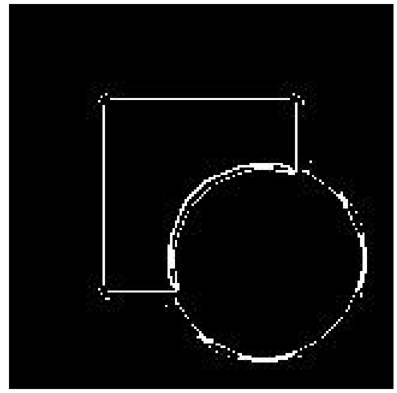

Figure 6. Result of constant diffusion coefficient FitzHugh-Nagumo edge detector

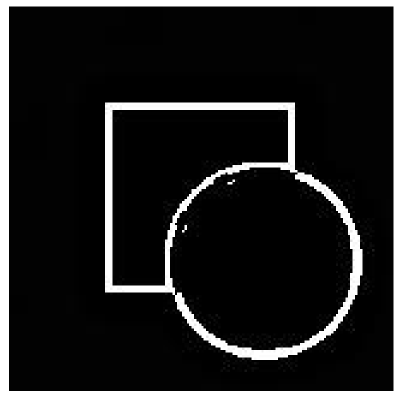

Figure 7. Result of FitzHugh-Nagumo edge detector with variable diffusion coefficient

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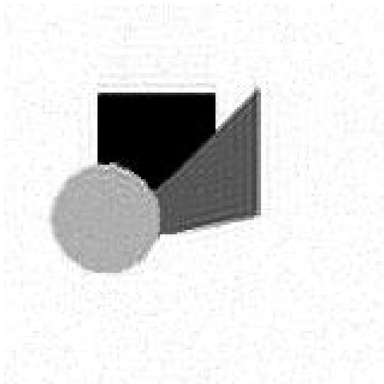

Figure 8. Source image with noise and ringing

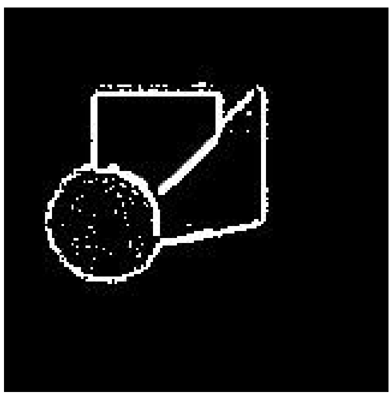

Figure 9. Result of FitzHugh-Nagumo edge detector with $D_u = 1.5$

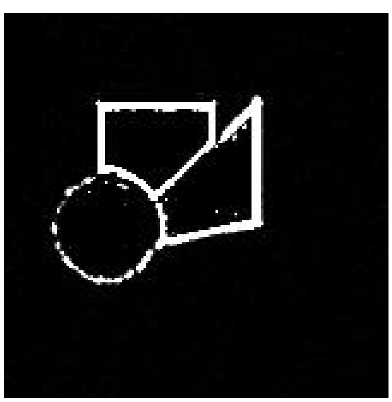

Figure 10. Result of FitzHugh-Nagumo edge detector with variable diffusion coefficient $D_u(x, y) = \frac{1}{1+\lambda^2|\bigtriangledown I(x,y)|^2}$