

Usage of Local Estimations and Object Spectral Representation at the Solution of Global Illumination Equation

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Abstract

The solution simulation of the global illumination equation by the local estimation of the Monte Carlo method is analyzed in the paper. Local estimation methods allow implementing simulation at the arbitrary reflectance law and directly calculating luminance in every point and direction of 3D scene. The local estimations are more effective methods than straight modeling. Their usage does not demand the mesh creation for the illumination map formation. The possibility of 3D object representation on the basis of the spherical harmonics is also considered in the paper. The spectral representation possesses the essential advantages for some object classes. The algorithms of the local and double local estimation with the usage of object representation on the basis of the spherical harmonics are implemented in the framework of paper. This algorithm allow realizing the physically adequate modeling of the global illumination equation solution and calculating the luminance directly that opens new horizons both in the photometry and computer graphics.

Keywords: *Global Illumination, Monte Carlo, Local estimation.*

1. INTRODUCTION

3D scene simulation is inherently the calculation of object luminance on the basis of the solution of the global illumination equation, which is an integral equation of the second order [1]:

$$L(\mathbf{r}, \hat{\mathbf{l}}) = L_0(\mathbf{r}, \hat{\mathbf{l}}) + \frac{1}{\pi} \int L(\mathbf{r}', \hat{\mathbf{l}}') \sigma(\mathbf{r}; \hat{\mathbf{l}}, \hat{\mathbf{l}}') \left| (\hat{\mathbf{N}}, \hat{\mathbf{l}}') \right| d\hat{\mathbf{l}}' \quad (1)$$

where $L(\mathbf{r}, \hat{\mathbf{l}})$ is the luminance at the point \mathbf{r} in the direction $\hat{\mathbf{l}}$, $\sigma(\mathbf{r}; \hat{\mathbf{l}}, \hat{\mathbf{l}}')$ is bidirectional scattering distribution function (reflection or transmittance), L_0 is the luminance of the direct radiation straight from sources.

The equation (1) has no analytical solution at an arbitrary law of reflection, so the methods of numerical simulation are used. The Monte Carlo methods, direct and reverse ray tracing are the most popular among them.

The reverse ray tracing suffers from basic defects connected with that the sizes of light sources as a rule are sufficiently small and the expectancy of hitting in the source is very low.

In case of the direct simulation the scene is divided into elements, in which the photon hits are calculated. As a result the illumination map is created. This approach is connected with the complications of mesh generation and the excessive consumption of memory.

The finite element method, which is called in the global illumination theory as a radiosity [3], is used for the solution of equation (1) on the basis of diffuse reflection approximation. In this case the equation (1) could be rewritten as [3]

$$M(\mathbf{r}) = M_0(\mathbf{r}) + \frac{\sigma}{\pi} \int_{\Sigma} M(\mathbf{r}') F(\mathbf{r}, \mathbf{r}') \Theta(\mathbf{r}, \mathbf{r}') d^2\mathbf{r}', \quad (2)$$

where $M(\mathbf{r})$ is the luminosity of the surface in point \mathbf{r} , $M_0(\mathbf{r})$ is the luminosity of the surface in point \mathbf{r} straightforward from sources,

$\Theta(\mathbf{r}, \mathbf{r}')$ is the visibility function of element $d^2\mathbf{r}'$ from point \mathbf{r} ,

$$F = \frac{\left| (\hat{\mathbf{N}}(\mathbf{r}), (\mathbf{r} - \mathbf{r}')) \right| \left| (\hat{\mathbf{N}}(\mathbf{r}'), (\mathbf{r} - \mathbf{r}')) \right|}{(\mathbf{r} - \mathbf{r}')^4}$$

is the infinitesimal form-factor.

In our paper we propose to use the local estimation of the Monte-Carlo method, which are well-known in the optics of atmosphere at the solution of radiative transfer equation [2, 4]. Rendering 3D objects represented by the spherical harmonics spectrum is analyzed in the paper.

2. LOCAL ESTIMATION

Equation (1) is not convenient for the statistic simulation, as the unknown function is under the integral in the point \mathbf{r}' , but it is determined in the point \mathbf{r} . In addition to that the variables \mathbf{r}' and $\hat{\mathbf{l}}'$ are not independent, and combined by the relation

$$\hat{\mathbf{l}}' = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (3)$$

Accordingly we can write down the equation (1) in the following form

$$L(\mathbf{r}, \hat{\mathbf{l}}) = L_0(\mathbf{r}, \hat{\mathbf{l}}) + \frac{1}{\pi} \int L(\mathbf{r}', \hat{\mathbf{l}}') \sigma(\mathbf{r}; \hat{\mathbf{l}}, \hat{\mathbf{l}}') \delta\left(\hat{\mathbf{l}}' - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\right) \frac{\left| (\hat{\mathbf{N}}, \hat{\mathbf{l}}') \right| \left| (\hat{\mathbf{N}}', \hat{\mathbf{l}}') \right|}{(\mathbf{r} - \mathbf{r}')^2} d^2\mathbf{r}'. \quad (4)$$

This expression contains the δ -function impeded the simulation by the local estimation of the Monte Carlo method. It is possible to eliminate the singularity in the expression (4) integrating over space. In addition to that the equation (2) contains no singularities and fit to simulation by the local estimation. Consequently the estimation for the arbitrary functional I_φ of $L(\mathbf{r}, \hat{\mathbf{l}})$ takes a form

$$I_\varphi = M \sum_{n=0}^{\infty} Q_n k(\mathbf{r}, \mathbf{r}'), \quad (5)$$

where $k(\mathbf{r}, \mathbf{r}')$ is the kernel of (4), Q_n is the weight of Markov chain, and M is the mean operator by the different random ray trajectories.

Markov chain in this case represents the sample of random ray trajectory, which is created using $L_0(\mathbf{r}, \hat{\mathbf{l}})$ as an initial probability and $k(\mathbf{r}, \mathbf{r}')$ as a transition probability [2, 4]. From every knots of this random ray trajectory $\mathbf{r} \square$, which coincide with the intersection points of ray with object surfaces, is calculated the contribution in the illumination in the given point \mathbf{r} by the term $k(\mathbf{r}, \mathbf{r}')$ in expression (5).

Of course all the probability distribution must satisfy to the normalizing condition. The distinctions of $L_0(\mathbf{r}, \hat{\mathbf{l}})$ and $k(\mathbf{r}, \mathbf{r}')$ from this condition generate the statistical weights Q_n . For example, in case of the diffuse reflection it is equal to the surface reflectivity.

The expression (5) was called the local estimation of the Monte Carlo method [2, 4]. It allows estimating the illumination in the given point. Thereby to calculate the illumination in some given

point \mathbf{r} it is necessary to create the Markov chain in the space, and at every point to calculate the value $k(\mathbf{r}, \mathbf{r}')$ for investigated points. The mathematical expectation of this value is equal an illumination.

Let's note that as opposed to the classical ray tracing the local estimation allows estimating in several points at once by one statistical sampling ray that sufficiently accelerates the convergence of calculations.

3. ACCURACY EVALUATION OF LOCAL ESTIMATION

The accuracy estimation of algorithm is possible on the basis of comparison with other known methods or with exact analytical solution of the equation (2) for some special situation. It is known two exact analytical solution of the global illumination equation: for the Ulbricht sphere and for so called Sobolev problem [9]. In case of the Ulbricht sphere we have the uniform distribution of the luminosity over sphere that makes this case ineffective for the accuracy estimation. In the Sobolev problem we solve the (2) for the luminosity distribution in case of the illumination of two parallel diffuse reflecting planes by the point isotropic source located between these planes. In this case the equation (2) is transformed to the set of two integral equations (for the first plane):

$$E_1(\mathbf{r}) = \frac{\rho_2}{\pi} \int \frac{E_2(\mathbf{r}') d^2 \mathbf{r}'}{[1 + (\mathbf{r} - \mathbf{r}')^2]^2} + \frac{h_1}{(h_1^2 + \mathbf{r}^2)^{3/2}}, \quad (6)$$

where $E_i(\mathbf{r})$ is the illuminance of i -th plane ($i=1,2$), ρ_i is the reflection coefficient of i -th plane, h_i is the distance from the source to the i -th plane. We assume the source intensity equal to 1 and $h_1 + h_2 = 1$.

For the solution of the equation set (6) let's execute the Fourier transformation of each equation that results in the set of linear algebraic equations. After the solution of the obtained equation set and inverse Fourier transformation we get the following expression:

$$E_1(r) = \frac{h_1}{(h_1^2 + r^2)^{3/2}} + \rho_2 \int_0^\infty \frac{e^{-h_1 k} \rho_1 k K_1(k) + e^{-h_2 k}}{1 - \rho_1 \rho_2 k^2 K_1^2(k)} K_1(k) J_0(kr) k^2 dk, \quad (7)$$

where $K_1(k)$ is the modified Bessel function of second kind, $J_0(kr)$ is the Bessel function of first kind.

In Figure 1 we can see the numerical comparison of the luminosity distribution calculation by the local estimation method and the expression (7).

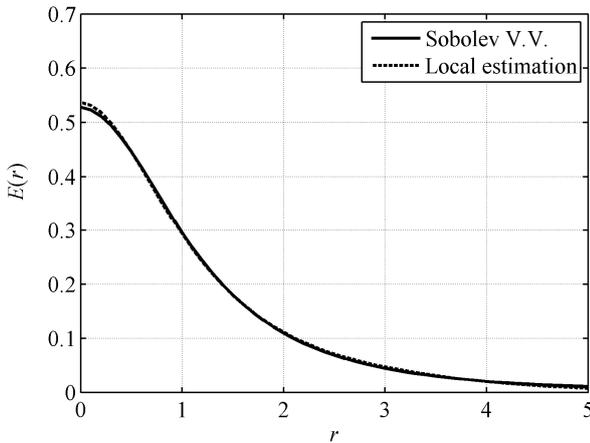


Figure 1. Illuminance distribution in case of the Sobolev problem: $h_1 = h_2 = 0.5$, and reflection coefficients $\rho_1 = \rho_2 = 0.5$.

In this case it was used 2000 rays at the calculation by the local estimation method, and the running time was less than 1 second for the computer AMD Athlon 64 X2 5200.

We realized also the solution of the Sobolev problem by the radiosity, and compared it with the local estimation: the local estimation exceeds the radiosity in computation speed at 80-90 times for the equal calculation accuracy.

4. DOUBLE LOCAL ESTIMATION

The equation of global illumination can be represented in the operator form

$$L = L_0 + \mathbf{K}L. \quad (8)$$

The solution of this equation is represented in the form of the Neumann series that allows conducting following transformations

$$\begin{aligned} L &= \sum_{n=0}^{\infty} \mathbf{K}^n L_0 = L_0 + \mathbf{K}L_0 + \sum_{n=2}^{\infty} \mathbf{K}^n L_0 = \\ &= L_0 + \mathbf{K}L_0 + \mathbf{K}^2 \sum_{n=0}^{\infty} \mathbf{K}^n L_0 = L_0 + \mathbf{K}L_0 + \mathbf{K}^2 L, \end{aligned} \quad (9)$$

that results in normal a form

$$\begin{aligned} L(\mathbf{r}, \hat{\mathbf{l}}) &= L_0(\mathbf{r}, \hat{\mathbf{l}}) + \mathbf{K}L_0 + \frac{1}{\pi^2} \int \sigma(\mathbf{r}; \hat{\mathbf{l}}', \hat{\mathbf{l}}) \delta\left(\hat{\mathbf{l}}' - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\right) F(\mathbf{r}, \mathbf{r}') \times \\ &\times \int_{(2)} L(\mathbf{r}'', \hat{\mathbf{l}}'') \sigma(\mathbf{r}'; \hat{\mathbf{l}}'', \hat{\mathbf{l}}') \delta\left(\hat{\mathbf{l}}'' - \frac{\mathbf{r}' - \mathbf{r}''}{|\mathbf{r}' - \mathbf{r}''|}\right) F(\mathbf{r}', \mathbf{r}'') d^3 r'' d^3 r'. \end{aligned} \quad (10)$$

The local estimation corresponding expression (10) can be called as double local estimation [2, 4] and takes the following view

$$I_\varphi = M \sum_{n=0}^{\infty} Q_n k(\mathbf{r}, \hat{\mathbf{l}}, \mathbf{r}', \hat{\mathbf{l}}'), \quad (11)$$

where

$$k(\mathbf{r}, \hat{\mathbf{l}}, \mathbf{r}', \hat{\mathbf{l}}') = \frac{1}{\pi^2} \sigma(\mathbf{r}; \hat{\mathbf{l}}'', \hat{\mathbf{l}}) \sigma(\mathbf{r}'; \hat{\mathbf{l}}', \hat{\mathbf{l}}'') F(\mathbf{r}', \mathbf{r}''). \quad (12)$$

In the expression (12) the δ -function is eliminated as a result of integration, and independent variables \mathbf{r} , $\hat{\mathbf{l}}$, \mathbf{r}' , $\hat{\mathbf{l}}'$, \mathbf{r}'' , $\hat{\mathbf{l}}''$ correspond to the geometry of ray propagation [4].

Thereby the double local estimation allows modeling the global illumination equation solution and calculating straight the luminance of incident radiation in the given point for the reflectance of orders higher than the first taking into account an arbitrary reflectance law. The first order can be calculated directly.

We do not know currently any software, which is capable to calculate the luminance directly. Thereby the double local estimation for the first time in light engineering and computer graphics allows computing the spatial-angular distribution of luminance in the given point of space.

5. SPECTRAL REPRESENTATION OF OBJECTS

Standard representation for the 3D objects is the mesh representation, in which objects are described by the set of vertices connected into faces. Such representation is universal and allows describing any object. However it is not capable to reproduce exactly many objects, and in case of the essential error influence on the result the solid modeling can be applied, when objects are described analytically. SolidWorks and TracePro are the most known programs using the similar approach.

One of the challenging directions of the 3D object representation is the spectral representation on the basis of spherical harmonics [5] that is the object surface representation by the series of spherical harmonics. In the sequel we will use the term "spectral representation" for short. Such object representation gives a possibility

of the reproduction quality control. So objects located far from a camera can be reproduced with low quality. In terms of photometry the essential advantage of this approach is the continuous representation of surface normals without any approximation. Let's analyze the mathematical techniques underlying the method of spectral representation.

The spherical harmonic $\{Y_k^m(\theta, \varphi) : m \leq k\}$ is the special function defined on the unit sphere

$$Y_k^m(\theta, \varphi) = (A_{km} \cos m\varphi + B_{km} \sin m\varphi) P_k^m(\cos \theta) \quad (13)$$

where θ is the zenith angle $[0, \pi]$, φ is the azimuth angle $[0, 2\pi]$, A_{km} , B_{km} are some coefficients, $P_k^m(\cos \theta)$ is the associated Legendre polynomials. The major property of spherical harmonics is their orthogonality:

$$\int_0^{2\pi} \int_0^\pi Y_k^m(\theta, \varphi) Y_{k'}^{m'}(\theta, \varphi) \sin \theta d\theta d\varphi = \delta_{k,k'} \delta_{m,m'}, \quad (14)$$

where δ is the delta Kronecker symbol.

Owing to this property any twice continuously differentiable function can be expanded into uniformly and absolutely convergent series of spherical harmonics [6]

$$f(\theta, \varphi) = \sum_{k=0}^{\infty} \sum_{m=0}^k (A_{km} \cos m\varphi + B_{km} \sin m\varphi) P_k^m(\cos \theta), \quad (15)$$

where A_{km} , B_{km} are the Fourier coefficients determined by formulae

$$A_{km} = \frac{\int_0^{2\pi} \int_0^\pi f(\theta, \varphi) P_k^m(\cos \theta) \cos m\varphi \sin \theta d\theta d\varphi}{\|Y_k^m\|^2},$$

$$B_{km} = \frac{\int_0^{2\pi} \int_0^\pi f(\theta, \varphi) P_k^m(\cos \theta) \sin m\varphi \sin \theta d\theta d\varphi}{\|Y_k^m\|^2}, \quad (16)$$

and $\|Y_k^m\|^2$ is the norm determined by the expressions

$$\|Y_k^m\|^2 = \frac{2\pi \varepsilon_m (k+m)!}{2n+1 (k-m)!}, \quad \varepsilon_m = \begin{cases} 2, m=0 \\ 1, m>0 \end{cases} \quad (17)$$

The calculation of factorial is the substantial complication by the expansion. As a result it is the essential difference of coefficients resulting in the extra miscalculation. At the expansions of spherical harmonics it is more convenient to use the Schmidt polynomial determined as

$$Q_k^m(\mu) = \sqrt{\frac{(k-m)!}{(k+m)!}} P_k^m(\mu). \quad (18)$$

It is not difficult to see that at the Schmidt polynomial expansion there is no necessity to calculate the factorials for the norm and that essentially improved productivity of calculations. Therefore, any object defined unambiguously on radius from some point can be expanded by spherical harmonics.

6. RENDERING SPECTRAL OBJECTS

At the global illumination equation solution by the local estimation method it is necessary to calculate surface normal and to determine the cross point of ray with an object. In case of the mesh object representation this problem is not difficult, but in case of the spectral representation it becomes a hard one. We analyze the determination of normal to the given point of object defined in the basis of spherical harmonics.

The normal in the point $(r_0, \theta_0, \varphi_0)$ on the surface defined in the spherical coordinate system is equal the value of its gradient in this point

$$\text{grad}U = \frac{\partial U}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \varphi} \hat{\mathbf{e}}_\varphi. \quad (19)$$

For the function described the surface in the spherical coordinate system we can write down the expression

$$U \equiv \sum_{k=0}^{\infty} \sum_{m=0}^k (A_{km} \cos m\varphi + B_{km} \sin m\varphi) Q_k^m(\cos \theta) - r = 0. \quad (20)$$

Let's find partial derivatives in accordance with the equation (19). The derivative with respect to r

$$\frac{\partial U}{\partial r} = -1, \quad (21)$$

The derivative with respect to φ

$$\frac{\partial U}{\partial \varphi} = \sum_{k=0}^{\infty} \sum_{m=0}^k (-mA_{km} \sin m\varphi + mB_{km} \cos m\varphi) Q_k^m(\cos \theta) \quad (22)$$

The derivative with respect to θ

$$\frac{\partial U}{\partial \theta} = \sum_{k=0}^{\infty} \sum_{m=0}^k (A_{km} \cos m\varphi + B_{km} \sin m\varphi) \frac{\partial Q_k^m(\cos \theta)}{\partial \theta} \quad (23)$$

Consequently, it is necessary to calculate the derivative of the Schmidt polynomial. It is known the recurrent relation for the Legendre polynomial [7]

$$\sqrt{1-\mu^2} \frac{\partial P_{l,k}^m(\mu)}{\partial \mu} = -\frac{i}{2} \left[\sqrt{(k+m)(k-m+1)} P_{l,k}^{m-1}(\mu) + \sqrt{(k-m)(k+m+1)} P_{l,k}^{m+1}(\mu) \right], \quad (24)$$

that taking into account the relation between the Schmidt and Legendre polynomial [7]

$$P_{0,k}^m(\mu) = i^{-m} Q_k^m(\mu), \quad (25)$$

allows to receive after some transformations the final expression

$$\frac{\partial Q_k^m(\mu)}{\partial \theta} = \frac{1}{2} \left[\sqrt{(k-m)(k+m+1)} Q_k^{m+1}(\mu) - \sqrt{(k+m)(k-m+1)} Q_k^{m-1}(\mu) \right]. \quad (26)$$

It is necessary to analyze separately the case $m=0$. In this case taking into account a number of the relations [8] it is not difficult to result in

$$\frac{\partial Q_k^0(\mu)}{\partial \theta} = \sqrt{k(k+1)} Q_k^1(\mu). \quad (27)$$

Using the derived gradient in the local spherical coordinate system $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\varphi)$ it is necessary to represent it in the world Cartesian coordinate system. Taking into account some known relations ones can get the final expression for the normal to object surface represented by spherical coordinates

$$\text{grad}U = \hat{\mathbf{i}} \left(-\rho \sin^2 \theta \cos \varphi + \sin \theta \cos \theta \cos \varphi \frac{\partial U}{\partial \theta} - \sin \varphi \frac{\partial U}{\partial \varphi} \right) + \hat{\mathbf{j}} \left(-\rho \sin^2 \theta \sin \varphi + \sin \theta \cos \theta \sin \varphi \frac{\partial U}{\partial \theta} + \cos \varphi \frac{\partial U}{\partial \varphi} \right) + \hat{\mathbf{k}} \left(-\rho \sin \theta \cos \theta - \sin^2 \theta \frac{\partial U}{\partial \theta} \right). \quad (28)$$

The search of the point of ray intersection with the object represented by the spherical harmonics is also a nontrivial task. Let's consider the object represented relative to the origin of the Cartesian coordinate system and the ray

$$\mathbf{r} = \mathbf{r}_0 + \xi \hat{\mathbf{1}}. \quad (29)$$

The cosine of angle θ of the point of ray intersection with the surface can be written as

$$\cos \theta = \frac{(\hat{\mathbf{k}}, \mathbf{r})}{|\mathbf{r}|} = \frac{z_0 + l_z \xi}{\sqrt{r_0^2 + \xi^2 + 2\xi(\hat{\mathbf{l}}, \mathbf{r}_0)}}. \quad (30)$$

For the vector $\boldsymbol{\rho}$ we have accordingly

$$\boldsymbol{\rho} = \mathbf{r} - \hat{\mathbf{k}}(\hat{\mathbf{k}}, \mathbf{r}) = \mathbf{r}_0 + \xi \hat{\mathbf{l}} - \hat{\mathbf{k}}(z_0 + l_z \xi). \quad (31)$$

The cosine and sine of angle φ have the form

$$\begin{aligned} \cos \varphi &= \frac{(\hat{\mathbf{i}}, \boldsymbol{\rho})}{|\boldsymbol{\rho}|} = \frac{x_0 + l_x \xi}{\sqrt{(r_0 + \xi \hat{\mathbf{l}} - \hat{\mathbf{k}}z_0 - \hat{\mathbf{k}}l_z \xi)^2}}, \\ \sin \varphi &= \frac{(\hat{\mathbf{j}}, \boldsymbol{\rho})}{|\boldsymbol{\rho}|} = \frac{y_0 + l_y \xi}{\sqrt{(r_0 + \xi \hat{\mathbf{l}} - \hat{\mathbf{k}}z_0 - \hat{\mathbf{k}}l_z \xi)^2}}. \end{aligned} \quad (32)$$

Therefore we get the dependence of angles θ and φ on one variable ξ . The equality takes place in the point of intersection

$$f_{sg}(\theta(\xi), \varphi(\xi)) - \mathbf{r}_0 - \xi \hat{\mathbf{l}} = 0. \quad (33)$$

Solving the equation (33) relatively the parameter ξ we can find all the point of ray intersection with the object surface. In case of the global illumination equation solution for scene rendering we are interested only by the first intersection with the object. It can be found both the standard procedures, for example the bisection method, and more complex and fast algorithms.

7. REALIZATION OF LOCAL ESTIMATION METHOD AND RENDERING OBJECTS REPRESENTED BY SPHERICAL HARMONICS

We implemented the algorithms of the local and double local estimations with 3D object representation in the basis of spherical harmonics. In Figure 2 we see rendering of the simple 3D scene. It is the room represented by the mesh with one column and one omni light source. The head of man in the scene is represented by the surface harmonics. The initial mesh contains 32 654 edges. In Figure 1 it was used $N = 32$ spherical harmonics to play the head. In this case the rendering time increased by 40% in comparison with the visualization of the mesh, but the amount of stored information decreased in more than 1,000 times.

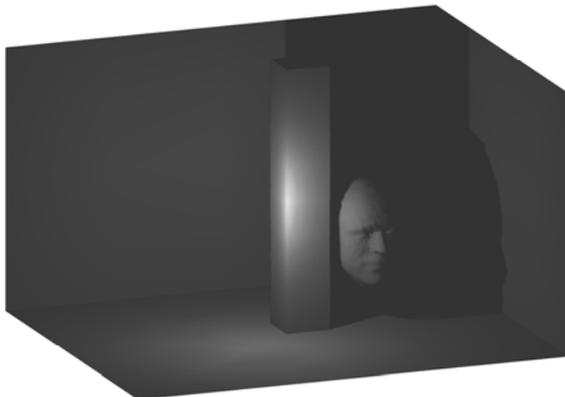


Figure 2. 3D scene rendering by the local estimation method.

8. CONCLUSIONS

The local estimation method of the Monte Carlo methods allows implementing the physically adequate simulation of the global illumination equation solution. In addition it allows obtaining directly the absolute values of luminance in every point and direc-

tion of scene. This algorithm opens the new horizons in the photometric research and can be the ground of the new rendering methods on the basis of direct solution of the global illumination equation.

The usage of the object spectral representation on the basis of spherical harmonics is the successful approach for the description of some class objects that allow reducing significantly the information content, which is necessary for the adequate reproduction. The possibility of continuous normal retrieval with the control accuracy in aggregate with the local estimation method is an important approach in the 3D scene rendering and its photometric investigation. The continuous restoring of normals has the special importance in case of luminance angular distribution calculations, where the wrong normal approximating results in considerable degradation of calculation accuracy.

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